

Frequency Spectra and Probability Distributions for Quantum Fluctuations

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Abstract This paper discusses two distinct, but related issues in quantum fluctuation effects. The first is the frequency spectrum which can be assigned to one loop quantum processes. The formal spectrum is a flat one, but the finite quantum effects can be associated with a rapidly oscillating spectrum, as in the case of the Casimir effect. This leads to the speculation that one might be able to dramatically change the final answer by upsetting the delicate cancellation which usually occurs. The second issue is the probability distribution for quantum fluctuations. It is well known that quantities which are linear in a free quantum field have a Gaussian distribution. Here it will be argued that quadratic quantities, such as the quantum stress tensor, must have a skewed distribution. Some possible implications of this result for inflationary cosmology will be discussed. In particular, this might be a source for non-Gaussianity.

1 Introduction

This paper will deal with two aspects of quantum fluctuations. One will be the frequency spectrum which is associated with Casimir energy, that is, with the expectation value of the stress tensor operator or other quadratic operators. Some earlier work will be reviewed, in which it was shown that the frequency spectra are wildly oscillating functions, which nonetheless have finite integrals. The finite Casimir energy always corresponds to a very small fraction of the area under one oscillation peak. The possibility of altering this remarkable cancellation will be discussed.

The second aspect of quantum fluctuations to be considered is the probability distribution associated with the fluctuations of smeared operators. The well-known result of a Gaussian distribution for linear operators will be rederived. It will then be shown by way of a simple example that the fluctuations of quadratic operators can be described by a skewed, and hence non-Gaussian, distribution. Some further comments and speculations on the generic form of the probability distribution for stress tensor fluctuations will be offered.

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2 Frequency Spectra

It is well-known that the formal frequency spectrum of quantum fluctuations is flat; all modes appear on an equal basis in the expansion of a quantum field operator. This leads to the formally divergent zero point energy

$$E_0 = \sum_{\lambda} \frac{1}{2} \hbar \omega. \tag{1}$$

However, this “unprocessed” spectrum is unobservable. The only quantities which can be observed arise from “processed” spectra, which have been modified by a physical process. The formation of a black hole, for example, processes the flat spectrum of incoming vacuum fluctuations and converts it into a Planck spectrum of outgoing particles, the Hawking radiation. The finite effects of one loop quantum processes may also be associated with a nontrivial frequency spectrum. Consider the Casimir effect as an example. The presence of boundaries modifies the divergent vacuum energy, (1), by a finite amount, the Casimir energy. For a scalar field which satisfies periodic boundary conditions in one of three spatial dimensions, this energy is (Units in which $\hbar = c = 1$ will be used in the remainder of this paper.)

$$E_C = -\frac{\pi^2 A}{90L^3}, \tag{2}$$

where L is the periodicity length and A is the transverse area.

It is of interest to ask whether one can assign a finite frequency spectrum to the Casimir energy, that is, find a function $\sigma(\omega)$ such that

$$E_C = \int_0^\infty d\omega \sigma(\omega). \tag{3}$$

This was done in Refs. [1, 2], where it was shown that one can obtain $\sigma(\omega)$ as a Fourier transform of the renormalized energy density operator at time-separated points. A more recent discussion is given in Ref. [3]. Let $T_{\mu\nu}(t - t')$ be the renormalized stress tensor operator evaluated at two points separated in time by $t - t'$. Then $T_{\mu\nu}(0)$, the operator evaluated at coincident points, is the observable stress-energy associated with the Casimir effect. In particular,

$$E_C = ALT_{tt}(0). \tag{4}$$

We can write

$$\sigma(\omega) = \frac{AL}{\pi} \int_{-\infty}^\infty e^{i\omega t} T_{tt} dt. \tag{5}$$

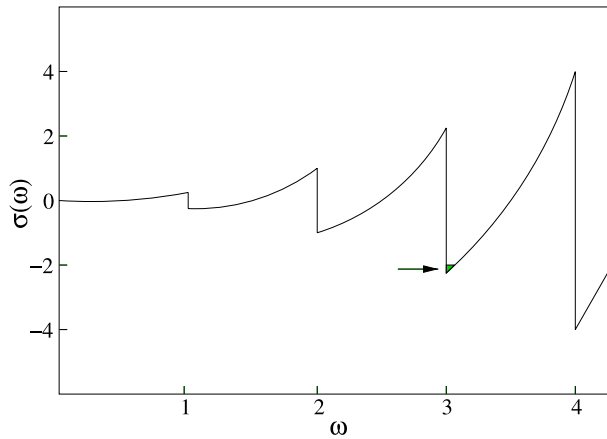
This function was evaluated explicitly for the electromagnetic case in Ref. [2], and for the case of a scalar field in Ref. [1]. In the latter case, the result is

$$\sigma(\omega) = -\frac{A\omega^2}{\pi L} S(\omega L), \tag{6}$$

where

$$S(x) = \sum_{n=1}^\infty \frac{\sin(nx)}{n} \tag{7}$$

Fig. 1 The frequency spectrum, $\sigma(\omega)$, given by (6) for the Casimir energy. The oscillations almost exactly cancel, leaving a net area under the curve equal to that of the shaded region indicated by the arrow. Here σ is in units of $2A/L$ and ω in units of $2\pi/L$



is a discontinuous, periodic function given by $S(x + 2\pi) = S(x)$ and

$$S(x) = \frac{1}{2}(\pi - x), \quad 0 \leq x < 2\pi. \tag{8}$$

At first sight, the integral of $\sigma(\omega)$ over all frequencies is poorly defined. However, it can be defined with a suitable convergence factor, such as an exponential function. One finds that the Casimir energy arises in the limit in which the convergence factor is removed. For example,

$$E_C = \lim_{\beta \rightarrow 0} \int_0^\infty d\omega e^{-\beta\omega} \sigma(\omega). \tag{9}$$

The result is that the contributions of different frequency intervals almost exactly cancel one another, leaving a finite result which is small compared to the area of each of the peaks in Fig. 1.

A similar spectrum can be associated with the asymptotic Casimir–Polder potential between a polarizable particle, such as an atom in its ground state, and a perfectly reflecting wall. In the limit that the atom is far from the plate, compared to the wavelength associated with the transition between the ground state and first excited state, Casimir and Polder [5] showed that the interaction energy is, in Gaussian units,

$$V_{CP} = -\frac{3\alpha_0}{8\pi z^4}, \tag{10}$$

where z is the distance to the wall, and α_0 is the static polarizability of the atom. It may be expressed as

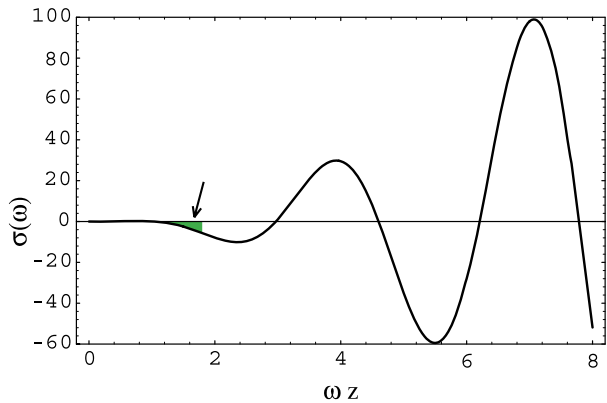
$$V_{CP} = \frac{\alpha_0}{4\pi z^3} \int_0^\infty d\omega \sigma(\omega), \tag{11}$$

where, in this case,

$$\sigma(\omega) = [(2\omega^2 z^2 - 1) \sin 2\omega z + 2\omega z \cos 2\omega z]. \tag{12}$$

As before, the integral on ω may be defined using a convergence factor. The frequency spectrum in this case is illustrated in Fig. 2.

Fig. 2 The frequency spectrum, $\sigma(\omega)$, for the Casimir–Polder potential. The oscillations again almost exactly cancel, and the net area is equal to that of the shaded region indicated by the arrow



As we have seen, the frequency spectra associated with Casimir-type effects can be wildly oscillatory, and yet tend to integrate to a small net energy. An obvious question is whether it may be possible to slightly modify the contribution of specific frequency intervals and upset the delicate cancellation between the different positive and negative peaks in Figs. 1 and 2. For example, if one could alter the reflectivity of parallel plates just in a selected frequency range, it would seem to be possible to have a Casimir force much larger than that between perfectly reflecting plates, and which could be repulsive as well as attractive [4]. A large, repulsive Casimir force could have significant industrial applications, such as allowing nearly frictionless bearings. Unfortunately, it is not at all straightforward to enhance Casimir forces. One might replace perfectly reflecting plates by dielectric slabs, which will have a finite, frequency dependent reflectivity. However, the Lifshitz theory [6] predicts that the force between two dielectric half-spaces will always be attractive and smaller in magnitude than the force in the perfectly reflecting limit. At least part of the reason for this is that the dielectric function $\varepsilon(\omega)$ must satisfy the Kramers–Kronig relations, which follow from the requirement of analyticity in the upper-half ω plane. This analyticity property is in turn a consequence of causality [7]. Analyticity constrains $\varepsilon(\omega)$ in such a way that the contributions of various frequency regions continue to cancel efficiently. If one were to allow non-analytic dielectric functions, then it becomes possible to construct situations in which the Lifshitz theory predicts repulsive forces [4].

Another approach to attempt to enhance Casimir forces was taken in Ref. [8]. There the force due to the quantized electromagnetic field on a small dielectric sphere near a perfectly reflecting plate was calculated. A quasi-oscillatory result was obtained, which can be either attractive or repulsive, and is much larger in magnitude than the Casimir–Polder force, from (10), for a non-dispersive sphere. In Ref. [9], it was shown that the qualitative form of the result does not depend upon whether the plate is perfectly reflecting or not. In both cases, the sphere was taken to have a plasma model dielectric function, and the resulting force oscillations as a function of distance from the plate have a scale set by the associated plasma wavelength. One can think of the effect of the sphere’s nontrivial frequency response as upsetting the cancellations that occur for a non-dispersive sphere, and which are illustrated in Fig. 2.

If the vacuum modes of the electromagnetic field gave the only contribution to the force on the sphere, then there would be a large repulsive force on the sphere at certain separations which would be large enough to levitate the sphere in the earth’s gravitational field, and should be observable. However, there is another contribution to the net force coming from

quantum mechanical fluctuations of the electric charge within the sphere. This is the effect of what is sometimes called the plasmonic modes. It was suggested by Barton [10] that the effect of the plasmonic modes should cancel the quasi-oscillatory terms coming from the vacuum modes. This suggestion is in fact correct, and was recently confirmed by an explicit calculation [11]. There is still a nonzero net force, but it is always attractive and of the order of magnitude of that for a non-dispersive sphere. The cancellation occurs only if the sphere plasmon modes are in their ground state. If they are excited, then the quasi-oscillatory behavior appears in the net force. This is similar to the case of an atom. If the atom is in its ground state, then the force is attractive at all separations [5]. If the atom is in an excited state, however, the net force is quasi-oscillatory and typically much larger in magnitude than in the ground state [12]. It is still unclear why the plasmon modes can cancel the effects of the vacuum modes so efficiently, and whether one can engineer materials with repulsive Casimir forces without exciting plasma oscillations.

3 Probability Distributions

In this section, we will turn to a somewhat different measure of vacuum fluctuations, the probability distribution associated with the fluctuations. First consider the case of a quantity which is linear in a free quantum field. It is well known that such quantities have a Gaussian distribution of probabilities. However, it will be worthwhile illustrating this result with an explicit example. Let $\varphi(x)$ be a free scalar field and let

$$\bar{\varphi} = \int \varphi(x) dV \tag{13}$$

be a smeared field operator averaged over some spacetime region. The dimension of the spacetime is not relevant here. We could also include a sampling function $f(x)$ and write $f(x)dV$ in place of dV without changing the result. Next consider the expectation value of a power of $\bar{\varphi}$ in the vacuum state. All of the odd powers have vanishing expectation value because the n -point functions vanish for all odd n . That is,

$$\langle \bar{\varphi}^{2n+1} \rangle = \int \langle \varphi_1 \varphi_2 \cdots \varphi_{2n+1} \rangle dV_1 dV_2 \cdots dV_{2n+1} = 0, \tag{14}$$

where $\varphi_1 = \varphi(x_1)$, etc. However, all of the even moments are nonzero and can be expressed as powers of the second moment,

$$\langle \bar{\varphi}^2 \rangle = \int \langle \varphi_1 \varphi_2 \rangle. \tag{15}$$

For example, the identity

$$\langle \varphi_1 \varphi_2 \varphi_3 \varphi_4 \rangle = \langle \varphi_1 \varphi_2 \rangle \langle \varphi_3 \varphi_4 \rangle + \langle \varphi_1 \varphi_3 \rangle \langle \varphi_2 \varphi_4 \rangle + \langle \varphi_1 \varphi_4 \rangle \langle \varphi_2 \varphi_3 \rangle \tag{16}$$

leads to the result

$$\langle \bar{\varphi}^4 \rangle = 3 \langle \bar{\varphi}^2 \rangle^2. \tag{17}$$

This is a special case of the general result

$$\langle \bar{\varphi}^{2n} \rangle = (2n - 1)!! \langle \bar{\varphi}^2 \rangle^n. \tag{18}$$

The latter result can be obtained from the following counting argument. Wick’s theorem allows us to decompose the $2n$ -point function $\langle \varphi_1 \varphi_2 \cdots \varphi_{2n} \rangle$ into a sum of products of two-point functions, just as is illustrated in (16). All that we need to know is the number of terms in this sum. The first contraction of the $2n$ -point function contains $2n - 1$ terms, as we can select any field to start, and it then has $2n - 1$ partners with which it can be contracted. Similarly, the contraction of each $(2n - 2)$ -point function to $(2n - 4)$ -point functions contains $2n - 3$ terms, and so on, leading to the factor of $(2n - 1)!!$ in (18).

This result for the general even moment implies that the probability distribution is a Gaussian function

$$P(\bar{\varphi}) = \frac{1}{\sqrt{2\pi \langle \bar{\varphi}^2 \rangle}} e^{-2\bar{\varphi}^2 / \langle \bar{\varphi}^2 \rangle}, \tag{19}$$

which is confirmed by the facts that

$$\langle \bar{\varphi}^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x) dx \tag{20}$$

and that

$$\langle \bar{\varphi}^{2n} \rangle = \int_{-\infty}^{\infty} x^{2n} P(x) dx = (2n - 1)!! \langle \bar{\varphi}^2 \rangle^n. \tag{21}$$

The result that $P(\bar{\varphi})$ is a Gaussian function is responsible for the prediction of Gaussian fluctuations in inflationary cosmology. In most versions of inflation, density perturbations are linked to the quantum fluctuations of a scalar inflaton field, which is treated as a free field [13]. This leads to a Gaussian distribution of density fluctuations, which seems to be consistent with observation [14].

Let us now turn to the fluctuations of quadratic operators, such as the smeared stress tensor. Now there is no particular reason to expect the probability distribution to be symmetric, much less Gaussian. In fact, as we will see from an explicit example, the distribution is in general a skewed one. The example will involve a massive scalar field in two-dimensional Minkowski spacetime, but in the limit of a very small mass. The Hadamard function for this field is

$$G(x, x') = \frac{1}{2} \{ \varphi(x), \varphi(x') \} = -\frac{1}{4} N_0(m\sqrt{\Delta t^2 - \Delta x^2}), \tag{22}$$

where m is the mass and N_0 is a Neumann function. In the limit of small m , this function becomes

$$G(x, x') = -\frac{1}{4\pi} \ln[\mu^2(\Delta t^2 - \Delta x^2)], \tag{23}$$

where $\mu = e^\gamma m/2$ and γ is Euler’s constant. The commutator function, in the limit that $m \rightarrow 0$, is

$$G_C(x, x') = [\varphi(x), \varphi(x')] = \frac{i}{4} [\theta(\Delta x - \Delta t) - \theta(\Delta x + \Delta t)], \tag{24}$$

where $\theta(x) = 1$ for $x > 0$ and $\theta(x) = -1$ for $x < 0$. Note that the commutator function is finite for $m = 0$, whereas the Hadamard function has a logarithmic divergence in this limit. This is a well-known feature of the massless scalar field in two-dimensional Minkowski spacetime. This divergence can be removed by selecting a vacuum state which breaks Lorentz invariance [15], but here we will assume a small, nonzero mass.

Next consider the average of the normal ordered square of φ over a finite time interval

$$\bar{\varphi}^2 = \frac{1}{T} \int_0^T : \varphi(t)^2 : dt. \tag{25}$$

In this model, all of the field operators will be taken to be at the same point in space, so that $\Delta x = 0$. We wish to calculate the vacuum expectation value of various powers of $\bar{\varphi}^2$. Begin with the second moment,

$$\langle (\bar{\varphi}^2)^2 \rangle = \frac{1}{T^2} \int_0^T dt_1 \int_0^T dt_2 \langle : \varphi(t_1)^2 : : \varphi(t_2)^2 : \rangle = \frac{2}{T^2} \int_0^T dt_1 \int_0^T dt_2 \langle \varphi(t_1) \varphi(t_2) \rangle^2, \tag{26}$$

where the last step follows from Wick’s theorem. The unsymmetrized two-point function can be written as a sum of the Hadamard and commutator functions

$$\langle \varphi(t_1) \varphi(t_2) \rangle = G(t_1, t_2) + \frac{1}{2} G_C(t_1, t_2). \tag{27}$$

We can now evaluate the second moment explicitly and find

$$\langle (\bar{\varphi}^2)^2 \rangle = \frac{1}{2\pi^2} \ln^2(T\mu) \tag{28}$$

in the limit that $T\mu \ll 1$. In this limit, the Hadamard function part of the two-point function gives the leading contribution.

Next we turn to the third moment

$$\langle (\bar{\varphi}^2)^3 \rangle = \frac{1}{T^3} \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \langle : \varphi(t_1)^2 : : \varphi(t_2)^2 : : \varphi(t_3)^2 : \rangle, \tag{29}$$

which can be written as

$$\langle (\bar{\varphi}^2)^3 \rangle = \frac{8}{T^3} \int_0^T dt_1 \int_0^T dt_2 \int_0^T dt_3 \langle \varphi(t_1) \varphi(t_2) \rangle \langle \varphi(t_2) \varphi(t_3) \rangle \langle \varphi(t_1) \varphi(t_3) \rangle. \tag{30}$$

In the limit of small $T\mu$, this becomes

$$\langle (\bar{\varphi}^2)^3 \rangle = -\frac{1}{\pi^3} \ln^3(T\mu) = \left[\frac{1}{\pi} \ln\left(\frac{1}{T\mu}\right) \right]^3. \tag{31}$$

The third moment is positive $\langle (\bar{\varphi}^2)^3 \rangle > 0$, and of the same order of magnitude as the second moment in the sense that

$$\langle (\bar{\varphi}^2)^3 \rangle^{\frac{1}{3}} = \sqrt{2} \langle (\bar{\varphi}^2)^2 \rangle^{\frac{1}{2}}. \tag{32}$$

Thus the probability distribution is significantly skewed.

The corresponding calculations for components of the quantum stress tensor have not yet been performed, but it is reasonable to guess that the results will be similar to those of this φ^2 model in two dimensions. The technical details are somewhat more complicated, however.

Let

$$\bar{T} = \int : T_{tt} : f(x) dV \tag{33}$$

be the average of the normal-ordered energy density over a spacetime region defined by the non-negative sampling function $f(x)$. The stress tensor correlation function is singular in the limit of null separated points,

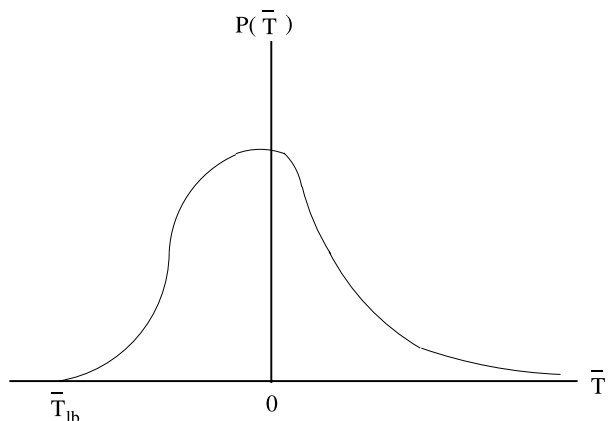
$$\langle :T_{tt}(x)::T_{tt}(x'):\rangle \sim (x - x')^{-2d}, \tag{34}$$

where d is the number of spacetime dimensions. Thus the integral for the second moment $\langle \bar{T}^2 \rangle$ will appear to contain a non-integrable singularity. However, it is possible to define such integrals using an integration by parts procedure [16–18]. It is also possible to use dimensional regularization [19]. In both approaches, the integral becomes finite. The third and higher moments can be defined in a similar manner. If $\langle \bar{T}^3 \rangle > 0$, as in the case of the φ^2 mode, then the probability distribution will again be skewed. There should be a connection between this distribution and the quantum inequalities which set lower bounds on the expectation value of \bar{T} in non-vacuum states [20–28]. In the case of temporal averaging only, these inequalities are of the form

$$\langle \bar{T} \rangle \geq \bar{T}_{lb} = -\frac{C}{\tau^d}, \tag{35}$$

where τ is the characteristic width of the sampling function and C is a constant. Thus negative energy densities are allowed in quantum field theory, but are tightly constrained. The lower bound in the expectation value in an arbitrary quantum state, and the lower bound on the probability distribution of fluctuations in the vacuum state are expected to coincide [29]. One way to understand this is to imagine writing the Minkowski vacuum state as a superposition of an alternative basis set of states. In each of these states, the measured \bar{T} is bounded below by \bar{T}_{lb} . Hence the possible fluctuations which can be observed in the Minkowski vacuum state are also bounded below by the same value. If the probability distribution $P(\bar{T})$ has positive skewness and a finite negative lower bound at $\bar{T} = \bar{T}_{lb}$, then we can make some general observations on $P(\bar{T})$. The first moment is defined to vanish, $\langle \bar{T} \rangle = 0$. A positive third moment implies a long tail in the positive direction. Thus the greater portion of the area will lie to the left of $\bar{T} = 0$. Thus a typical measurement of the sampled energy density is more likely to yield a negative than a positive value. However, when the measured value is positive, its magnitude is likely to be greater than when it is negative. Figure 3 is a sketch of a hypothetical probability distribution which satisfies all of these conditions. A more detailed analysis of the probability distribution for various cases is currently in progress [29].

Fig. 3 A possible probability distribution $P(\bar{T})$ for stress tensor fluctuations is illustrated as a function of an averaged stress tensor component \bar{T} . This distribution is zero below the lower bound, $\bar{T} < \bar{T}_{lb} < 0$, but has a nonzero tail extending infinitely far in the positive direction. The majority of the area under the curve lies in the negative region, $\bar{T} < 0$



The non-Gaussian nature of quantum stress tensor fluctuations may have applications to inflationary cosmology in the form of a source of a non-Gaussian component in the density perturbations. This is currently under investigation [30].

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